Notes on "A Result in Visual Aesthetics"

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I was general chair of OOPSLA in 2007, held in Montréal, Quebec. For the special event, I asked a talent agent to hire the least expensive local rock band that was not embarrassingly bad. That band's name as *Careless and Sloppy*. Somehow, reading this early paper by Christopher Alexander reminded me of them.

I am interested in the question of how reliable is Alexander's "mathy" support for his ideas. In "Notes on the Synthesis of Form" he implies that a computer program could produce a good decomposition of a design problem; and in fact he presents a derivation of the key part of that program and states that he actually wrote the program. He does *not* state that the program produced the shown solution. I spent a lot of time trying to reproduce the implied results, and could not come close. In 2022 I stumbled on the program Alexander did write, and it also could not come close.

In this paper, "Aesthetics," he implies that some formal analyses can help explore the question of aesthetics, and this note is my exploration of those analyses.

Montréal.

[NOTE TO THE READER: If you are not accustomed to reading papers typeset by LaTex, please keep in mind that the figures labeled **Floating-Figure** are placed in the document by a whimsical algorithm. They may look like they are embedded in sensible places in the text, but they likely are not. While reading, when you happen upon one, skip over it unless you were directed to it by a specific reference.]

This paper precedes Notes by a few years; it was published in 1960 (included as Appendix A). I had never heard of it before 2022. It was an early attempt by Alexander to understand how and why people sense beauty. Alexander conducted a series of experiments using abstract forms, asking participants to order those forms in various ways in order to discover some analytic, non-verbal characteristics related to how those subjects—and people—perceive form. The results of the experiments yielded some symbolic information, and Alexander examined and combined that information in various ways to glean some speculations.

1 TRIADS

When I encountered this paper in early 2022, I proceeded to analyze the claims made about the actual experiments. I am interested in his attention to detail, the accuracy of his analyses, and depth of his investigations—not his conclusions. The abstract says, in part:

> *Subjects were given eight forms and asked to sort them in a number of ways on the basis of overall similarity; they were also asked to state the order of their preferences among the forms.*

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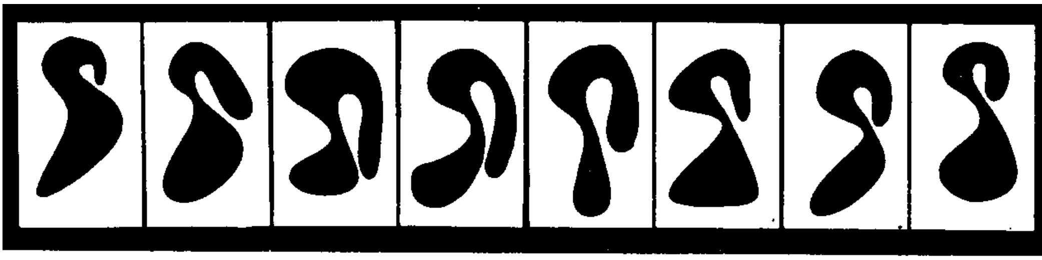

Fig. 1. The stimuli A, B, C, D, E, F, G, H.

The eight forms were given the names *A, B, C, D, E, F, G, H.* After a complex series of tasks, each subject ended up generating a number of *triads*, which roughly state preferences. These triads are derived from a set of tables for each subject, which represent sets of observed *perceptual distances.* Each subject was presented with pairs of figures on the table in front of them; then a series of other figures were presented, and the subject was asked to place each of these next to the original that seemed closer. The result was a set of tables like this one in Figure 1.

C	E
D	A
F	B
	G
	H

Floating-Figure 1. Sample Raw Data

This table of raw data (Figure 1) provides the following observed 6 perceptual distances for a subject, where *CD*, for example, represents the "perceptual distance" between figures *C* and *D*, as perceived by that subject:

$$CD < DE, CF < FE, EA < AC, EB < BC, EG < GC, EH < HC$$

Note that in this notation, $XY = YX$, so by convention I (and Alexander) will use alphabetic order in such pairs:

$$CD < DE, CF < EF, AE < AC, BE < BC, EG < CG, EH < CH$$

Now suppose we have two pieces of raw data (Figure 2). This gives us 12 perceptual distances:

$$CE < EF, CD < DF, CE < EF, AF < AC, FG < CG, FH < CH$$
$$CE < CF, DE < DF, EG < FG, AF < AE, BF < BE, FH < EH$$

C	F
B	A
D	G
E	H

and

E	F
C	A
D	B
G	H

Floating-Figure 2. Sample Pair Raw Data

Alexander says:

> *Each table like* [Figure 1] *is in fact a condensed statement of six inequalities among the perceptual distances: for since F is put under C rather than under E, the table indicates that CF < EF; and five other facts of the same kind. Similarly the tables* [Figure 2] *tell us that CE < EF and CE < CF. We may combine the three inequalities to give CE < CF < EF. (When it happened—as it did about once for every subject—that the three inequalities were inconsistent, then one of them was reversed; whichever one the subject had been most uncertain of, whichever one he had changed his mind about most often.) The statement CE < CF < EF tells us that in the triad CEF the perceptual distance EF is the greatest of the three, so we write the triad ECF. The position of C between E and F is most important and will be referred to as this triad's betweenness.*

Regarding betweenness, Alexander seems to be saying that it does not matter whether the triad is called ECF or FCE. He actually doesn't say that directly, but because all the triads he lists in Figure 3 happen to have their first and last elements in alphabetical order and because all his other stated examples of betweenness also have this property, I concluded it.

Alexander did not provide his original set of gathered tables, which would have provided more information to analyze for consistency. As it is, we have only the triads.

Alexander's original table of triads is in Figure 3.

2 ALEXANDER'S DIMENSIONS

The first thing Alexander tries is a kind of factor analysis, which is used to uncover unobserved or hidden variables that account for observed behavior. In this case, he tries to come up with a set of *dimensions* that combine to explain the triads. Here is the process he used:

> (1) *Select, by inspection of the data, that pair which seems to be most dissimilar (where the perceptual distance is the greatest), and use this pair as end-points of the first dimension.*
> (2) *Order the remaining six letters between these end-points so as to accommodate as many of the fifty-six triads as possible. That is to say, construct that order which preserves betweenness for as many of the triads as possible.*
> (3) *Repeat the above procedure for those triads not accommodated by the first dimension. If these are not enough to define a second dimension uniquely, construct that one which overlaps the first as little as possible. (It is inevitable, in spite of this, that there will be some triads whose betweenness is satisfied on both dimensions.)*
>
> *In principle the extraction of dimensions should go on in this way until all the triads have been accommodated. But, in fact, no more than two dimensions were ever needed. All the triads, bar one or two, were accommodated by the first two dimensions extracted.*

Subject 1	Subject 2	Subject 3	Subject 4	Subject 5	Subject 6
ABC ABD	ABC ABD	ABC ABD	ABC ABD	ABC ABD	ABC ABD
ABE ABG	ABE ABF	ABE ABH	ABE ABF	ABE ACD	ABE ABF
ABH ADC	ABH ACE	ACE ADC	ABH ADC	AEC AED	ABH ADC
ADE ADF	ADC ADE	ADE AFB	ADE AEC	AFB AFC	ADE AEC
ADH AEC	ADF AFC	AFC AFD	AFC AFD	AFD AFE	AFC AFD
AFB AFC	AGB AGC	AFE AGB	AFE AFH	AFG AFH	AFE AFH
AFE AGC	AGD AGE	AGC AGD	AGB AGC	AGB AGC	AGB AGC
AGD AGE	AGF AGH	AGE AGF	AGD AGE	AGD AGE	AGD AGE
AGF AGH	AHC AHD	AHC AHD	AGF AGH	AHB AHC	AGF AGH
AHC AHE	AHE AHF	AHE AHF	AHC AHD	AHD AHE	AHC AHD
AHF BDC	BCD BCE	AHG BDC	AHE BDC	AHG BCD	AHE BDC
BDE BDF	BCF BCH	BDE BEC	BEC BED	BEC BED	BDE BDG
BDG BEC	BDE BGD	BFC BGF	BFC BGC	BGC BHG	BDH BEC
BFC BFE	BGE BHF	BGH BHF	BGD BGF	CBF CBH	BFC BFE
BFH BGC	BHG CBG	CBG CBH	BGH BHC	CDE CDG	BFH BGE
BGE BGF	CDG CHF	CDE CDH	BHE BHF	CEG CEH	BGF BHE
BHC BHE	CHG DBF	CFG CFH	CDE CDG	CFG CGH	BHG CBG
CDG CED	DBH DCE	CGH DBF	CDH CEG	CHF DBF	CBH CDG
CEF CEG	DCF DCH	DBG DBH	CFG CFH	DBG DBH	CEG CEH
CEH CFG	DGF DGH	DCF DCG	CHG DBF	DCF DCH	CFG CGH
CHG DBH	DHF EAF	DGF DGH	DBH DCF	DEF DEG	CHF DBF
DCF DCH	EBF EBH	DHF EBF	DEH DFH	DEH DGF	DCE DCF
DEF DFH	ECF ECG	EBG EBH	DGF DGH	DGH DHF	DCH DEH
DGF EDG	ECH EDF	ECF ECG	EBF EBG	EBF EBG	DGE DGF
EDH EFG	EDG EDH	ECH EDF	ECF ECH	EBH ECF	DGH DHF
EFH EHG	EGF EGH	EDG EDH	EDF EDG	EGF EGH	ECF EDF
FCH GBH	EHF FBG	EFH EGF	EGF EHF	EHF FBG	EGF EHF
GDH GFH	FCG FHG	EGH FGH	EHG FGH	FBH FHG	EHG FHG

Floating-Figure 3. Original CA Triads

When you look at steps 1 and 2 together, you get the idea that step 1 is a shortcut of sorts: there are 720 permutations of 6 letters, but 40,320 of 8. In 1959, a computer would have trouble with that many permutations, so Alexander "inspected" the triads to determine the first and last letters for a dimension. My computer laughs at 40,320 permutations, so I tried both his way and the exhaustive way. Later I did some more exploration about the so-called "inspection."

For an exhaustive search, I combined steps 1 and 2 like this:

> *Order the eight letters so as to accommodate as many of the fifty-six triads as possible. That is to say, construct that order which preserves betweenness for as many of the triads as possible.*

Let's first look at the dimensions Alexander came up with (Figure 4).

Subject	Dimension 1	Dimension 2	Leftovers_R	Leftovers_U	Preference Order
Subject 1:	ABGDHFEC	EDGCBFH*	(3)	[4]	CADGEBFH
Subject 2:	ABFGHDCE	EADGBCHF[1]	(2)	[2]	HGAFEBDC
Subject 3:	AHGFBCDE[2]	ABDEGHFC	(1)	[1]	BGHFDCEA
Subject 4:	ABGHFEDC[3]	EDBGACHF	(2)	[2]	GCAHDEBF
Subject 5:	AFHGBECD[4]	ECDFBHG*	(2)	[2]	DBCEGHFA
Subject 6:	AGBFHDEC[5]	BDCGEHF*	(5)	[5]	HDCBFGEA

Floating-Figure 4. Original CA Dimensions

The dimensions are as described by Alexander above. The numeric superscripts correspond to overlaps with the outcomes of my version of his process (shown later). The asterisks indicate that the second dimension does not contain all the figure names because the first dimension accommodated all the triads that mention the missing names. Alexander says of these in a footnote, "for three subjects the first dimension accommodated all triads containing A. In these cases the second dimension does not contain A.”

The Leftovers indicate how many triads have been left over after generating both dimensions. $\text{Leftovers}_{\text{U}}$ is the number left over after using exactly the triads in Figure 3—the so-called *unrepaired triads.* Preference Order is as determined by subjects using pairs of figures during the experiment.

Based on the footnote mentioned, we can figure out what “accommodate” means. Alexander says that the Dimension 1 for Subject 1 accommodates all the triads containing A, which implies that it accommodates ABC. A dimension accommodates a triad xyz if the dimension is of the form *x*y*z*—that is, x, y, & z appear in that order in the dimension.

A dimension seems to be a projection of a model onto a linear order. That is, we can think of a 2-dimensional space containing the figures, and Dimension 1 is a projection of the figures on one axis, and Dimension 2 on the other.

Let’s take a look at how well Alexander’s dimensions accommodate the triads. See Figures 5&6. We’ll explain the column for Subject 1. At the top we see the dimensions ABGDHFEC & EDGCBFH. Under ABGDHFEC are the triads that are not accommodated by that dimension. Note that it includes AFB, which contradicts the statement “for three subjects the first dimension accommodated all triads containing A.” Under EDGCBFH we see the triads that not accommodated by the pair of dimensions—hence “leftovers.” The number of them correspond to the numbers under $\text{Leftovers}_{\text{U}}$ in Figure 4. For Subject 1, four are left over and for Subject 6 five are left over. This contradicts Alexander’s statement “all the triads, bar one or two, were accommodated by the first two dimensions extracted.”

I suspect none of these computations were done with a computer, and I found it mind-numbing to pore over the triads and dimensions, trying to count things and to take into account that a triad like CEF has the same accommodating effect as FEC (a triad for Subject 1 accommodated by ABGDHFEC). I believe Alexander suffered from the difficulty of dealing with data labeled like this.

Subject 1		Subject 2		Subject 3	
ABGDHFEC	EDGCBFH	ABFGHDCE	EADGBCHF	AHGFBCDE	ABDEGHFC
AFB	AFB	ADF	BCD	ABH	CDH
BDG	BDG	AGB	BHG	ADC	
BFH	DEF	AGF		BDC	
DBH	FCH	AHF		BEC	
DCF		BCD		BFC	
DCH		BCF		BGF	
DEF		BCH		BHF	
DFH		BHF		CDH	
DGF		BHG		DGF	
EDH		CBG		DHF	
FCH		DBF		EGF	
GBH		DBH			
GFH		DCF			
		DCH			
		DGH			
		EAF			
		EBF			
		EBH			
		EGH			
		FBG			
		FCG			
		FHG			

Floating-Figure 5. Unrepaired CA Dimensions Leftovers (Part 1)

Subject 4		Subject 5		Subject 6	
ABGHFEDC	EDBGACHF	AFHGBECD	ECDFBHG	AGBFHDEC	BDCGEHF
ADE	AFH	BGC	BGC	BDG	BHG
AFH	FGH	BHG	CDE	BDH	CBH
AGB		CDE		BGE	DBF
DBF		CDG		BGF	ECF
DBH		CFG		BHG	EGF
DCF		ECF		CBH	
DGF		FBG		CGH	
DGH		FBH		DBF	
EBF				DCE	
EBG				DCF	
ECF				DCH	
ECH				DEH	
EDF				DGE	
EDG				DGF	
EGF				DGH	
EHF				ECF	
FGH				EGF	
				FHG	

Floating-Figure 6. Unrepaired CA Dimensions Leftovers (Part 2)

3 RPG'S DIMENSIONS

Subject	Dimension 1	Dimension 2	Leftovers_U
Subject 1:	ABGDFHEC	AHCFEBDG	(3)
Subject 2:	EDCGBHAF	AGHBDCFE	(3)
	ECDGBHAF	AGHBCDF	(4)
	AFHBGDCE	EADGBCHF[1]	(3)
	AFHBGCDE	EAGDBCHF	(3)
Subject 3:	AHGFBCDE[2]	CEDABGHF	(1)
Subject 4:	ABGHFEDC[3]	AFHCGBDE	(1)
	ABGHFDEC	AFHCGBED	(2)
Subject 5:	AFHGBECD[4]	CDEGHBF	(2)
Subject 6:	ABFHGDEC	AGHEDBCF	(5)

Floating-Figure 7. rpg Two Dimensions (Unrepaired Triads)

I wrote a program that takes a subject's triads and tries to find dimensions. It works like this.

It takes the initial population of figures (A, B, C, D, E, F, G, H) and generates all permutations. There are 40,320 of them. For each of these it looks at how many triads are accommodated. It gathers all the ones with the same, maximum number of accommodated triads. For each of those, it removes the accommodated triads from the initial set and repeats the process. Because Alexander reported only two dimensions, the program stops here and tries to find the best pair of dimensions. It has all the best ones for the first dimension and, for each of those, the best ones for the second dimension. The program looks at all pairs of such dimensions and computes the number of overlaps.

Overlaps are computed for a sequence by looking at all the implied precedence pairs and counting the intersection. For example, given these two sequences: (A, B, C, D) and (A, C, B, D), the implied precedences are:

$$A < B, A < C, A < D, B < C, B < D, C < D$$
$$A < C, A < B, A < D, C < B, C < D, B < D$$

where $x < y$ means that x occurs before y in the list. Notice that they agree on $A < B, A < C, A < D, B < D, C < D$, which means the overlap is 5 out of 6. The dimension finder then collects all the pairs with the smallest overlap.

The result is shown in Figure 7; and as in the dimensions for Alexander, the count of unaccommodated triads is shown. The superscripts show agreement with Alexander's dimensions.

You will also notice that for Subjects 2 and 4 there are multiple pairs of dimensions; this is because the number of overlaps were the same.

Subject	Dimension 1	Dimension 2	Dimension 3
Subject 1:	ABGDFHEC	AHCFEBDG	EDGCFH
Subject 2:	AFHBGDCE	EADGBCHF^{1}	BCHGD
Subject 3:	AHGFBCDE^{2}	CEDABGHF	BFC
Subject 4:	ABGHFEDC^{3}	AFHCGBDE	FGH
Subject 5:	AFHGBECD^{4}	CDEFBHG	BGFCE
	AFHGBECD^{4}	CDGEHBF	ECFG
	AFHGBECD^{4}	EDCGHBF	CFDG
	AFHGBECD^{4}	CDEGHBF	ECFG
Subject 6:	ABFHGDEC	AGHEDBCF	DCFGBEH

Floating-Figure 8. rpg Three Dimensions (Unrepaired Triads)

The process for determining dimensions described above actually is a truncation of the full one that I use to find all dimensions. This means that at every stage, it computes unaccommodated triads and simply keeps going until there are no more or it can't improve (that never happened). It computes overall non-overlap by summing overlap over running pairs of dimensions. The results are in Figure 8.

Let's look at the first/last letter pairs for Alexander's first dimensions to see how well my program matches his "inspection of the data." For my 3-dimensional cases, all the first/last letter pairs agree for Dimension 1 with Alexander's. For my 2-dimensional cases, my Subject 2 dimension pairs have some mismatches: looking at the dimension pairs line by line for my Subject 2 dimensions, my line 1 / Dimension 2 matches first/last letters with Alexander's; my line 2 does not; and the last 2 lines match for first dimensions. But for Alexander's second dimension for Subject 2, each of my lines has one of the pair of dimensions matching first/last letters. This means that Alexander's "inspection of the data" worked well.

4 INCONSISTENT TRIADS

Alexander says this in a footnote:

> *Among the very few triads not accommodated by the first two dimensions, one or two are even inconsistent—Subject 1's AFB, for instance. Probably these minor vagaries are the result of the subject's indecision already discussed, and would be smoothed out if the subject were givens still longer opportunity to reach consistent choices.*

There is some evidence that Alexander addressed these inconsistencies: if you were to delete the triad AFB for Subject 1, his statement would be correct that "for three subjects the first dimension accommodated all triads containing A" (they would be Subjects 1, 5, & 6). But as it stands, Figures 5&6 show the contradiction.

After a little thought it occurred to me that an easy way to determine inconsistencies among the triads was to use topological sorting. Topological orderings are closely related to the concept of a linear extension of a partial order in mathematics.

For non-math people: If you have a set of precedence rules (that say that $x \leq y$ for some elements of a set), a topological sorting is a total order (a definite, deterministic sequence) that is consistent with those rules. A silly example: if I said "I don't care in what order people get their slices of pizza as long as I get mine before you," then any one of every possible order that has me before you is a topological sorting. Unlike sorting a list of distinct numbers into numeric order, topological sorting doesn't need to come out with the same order every time. As another example, suppose you had a room full of people and you asked them to sort themselves into the order of the calendar birthdays (only months and days, not years), then for any set of people with the same birthday, you would have to figure out a way to put those guys in a definite order (flip a coin, measure height, most poems written). If you didn't have a tie-breaking rule like that, then bunches of people with the same birthday could go in any order.

On the other hand, if it is not possible to find a way to satisfy all the precedence rules ("me before you"), then the rules are inconsistent.

Can we find rules for the triads? We can try to sort pairs of figures by how similar they are. If you have the triad ABC, you can set up these precedence rules:

$$AB < AC, BC < AC$$

again where $<$ means precedes. (Technically you might need to reverse those because you need $\leq$.) This pair of relationships means that the pair A and B are more perceptually similar to each other than A and C are to each other; and that B and C are more perceptually similar to each other than A and C. Then you can gather a set of rules derived from all the triads for a subject. Keep in mind that although Alexander used the phrase "perceptual distance," there is no measurable distance involved, only the notion of being more similar.

Each subject has 56 associated triads, which turn into 112 precedence rules. And to be explicit, we are looking to topologically sort the following pairs using the precedence rules derived from each subject's triads:

$$AB, AC, AD, AE, AF, AG, AH, BC, BD, BE, BF, BG, BH, CD, CE, CF, CG, CH, DE, DF, DG, DH, EF, EG, EH, FG, FH, GH$$

I wrote a program to collect the triad-based constraints and use them to try to topologically sort the above perceptual similarity pairs. I ran it on the triads reported by Alexander; only one was consistent: Subject 2's triads. For the others, I set out to find the minimal set of changes to "repair" the triad sets. It works like this:

Go one by one through the triads; throw that one out and try topologically sorting the precedences derived from the rest. If that doesn't find a bad triad, do the same thing for all pairs of triads, then all triples, etc. I never had to go beyond throwing out three.

When I found a bad set of triads, I would try altering them to see whether the result was consistent. For all five of the inconsistent triad sets, I found repairs that rendered the set consistent. They are in Figure 9.

Subject 1	Subject 2	Subject 3	Subject 4	Subject 5	Subject 6
AFB⟶ABF	-	ABH⟶AHB	CDH⟶HCD	BCD⟶BDC	CBH⟶CHB
CEH⟶ECH	-	-	-	CFG⟶CGF	BGF⟶BFG
-	-	-	-	-	CGH⟶CHG

Floating-Figure 9. Triad Repairs

I refer to these revised triads as "repaired triads," and generally I use them rather than Alexander's. (Note, using them instead of Alexander's doesn't make much difference, if any.)

The topological sort algorithm can be tweaked to show all the items that are equivalent—equivalent in the sense that in making a definite order from the partial order, if one member of the equivalent set could go next, any of them could. In the birthday sorting example, any one of the people with the birthday September 12 could be replaced by a different person with that birthday in a final order—they would be tied, as it were.

The ordered clusters of perceptual distance pairs is shown in Figures 10–15. Keep in mind that these clusters represent pairs of figures which seem to have the same distances (or, at least, indistinguishable distances) between them, as judged by the precedence ordering determined by the triads—even though there are no numeric distances available.

To show the limitations of using these clusters, I tried to use them to find dimensions in two different ways. The first program cycles through all the possible dimensions (40,320) and finds all the dimensions that could have been produced by the precedences that gave rise to the clusters. It found none.

The second program also cycles through the possible dimensions and finds the ones that best satisfy a similarity predicate that I devised. That predicate takes a dimension and produces a set of groups to compare to the clusters; it finds adjacent pairs, pairs separated by 1 letter, pairs separated by 2, etc. The resulting distance groups for the dimension ABGDFECH is shown in Figure 16.

$$\left\{\begin{matrix}\mathrm{AB}\\\mathrm{BD}\\\mathrm{CE}\\\mathrm{CH}\\\mathrm{DE}\\\mathrm{DG}\\\mathrm{EF}\\\mathrm{FG}\end{matrix}\right\} < \left\{\begin{matrix}\mathrm{BG}\\\mathrm{CD}\\\mathrm{CF}\end{matrix}\right\} < \left\{\begin{matrix}\mathrm{AG}\\\mathrm{DF}\\\mathrm{FH}\end{matrix}\right\} < \left\{\begin{matrix}\mathrm{AD}\\\mathrm{BF}\end{matrix}\right\} < \left\{\mathrm{BH}\right\} < \left\{\mathrm{DH}\right\} < \left\{\begin{matrix}\mathrm{EH}\\\mathrm{GH}\end{matrix}\right\} < \left\{\begin{matrix}\mathrm{AH}\\\mathrm{EG}\end{matrix}\right\} < \left\{\begin{matrix}\mathrm{AF}\\\mathrm{BE}\\\mathrm{CG}\end{matrix}\right\} < \left\{\begin{matrix}\mathrm{AE}\\\mathrm{BC}\end{matrix}\right\} < \left\{\mathrm{AC}\right\}$$

Floating-Figure 10. Subject 1 Perceptual Distance Constraints (Repaired Triads)

$$\begin{Bmatrix} \mathrm{AG} \\ \mathrm{BC} \\ \mathrm{CD} \\ \mathrm{CE} \\ \mathrm{CH} \\ \mathrm{DG} \\ \mathrm{FH} \\ \mathrm{GH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{BH} \\ \mathrm{CF} \\ \mathrm{DE} \end{Bmatrix} < \begin{Bmatrix} \mathrm{BF} \\ \mathrm{BG} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AB} \\ \mathrm{BD} \\ \mathrm{CG} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AH} \\ \mathrm{DH} \\ \mathrm{EG} \\ \mathrm{FG} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AD} \\ \mathrm{BE} \\ \mathrm{DF} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AF} \\ \mathrm{EH} \end{Bmatrix} < \{\mathrm{AC}\} < \{\mathrm{AE}\} < \{\mathrm{EF}\}$$

Floating-Figure 11. Subject 2 Perceptual Distance Constraints

$$\begin{Bmatrix} \mathrm{AH} \\ \mathrm{BD} \\ \mathrm{BG} \\ \mathrm{CD} \\ \mathrm{CF} \\ \mathrm{DE} \\ \mathrm{FG} \\ \mathrm{GH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AG} \\ \mathrm{BE} \\ \mathrm{BH} \\ \mathrm{CE} \\ \mathrm{FH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AF} \\ \mathrm{BF} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AB} \\ \mathrm{BC} \end{Bmatrix} < \{\mathrm{CG}\} < \{\mathrm{DG}\} < \begin{Bmatrix} \mathrm{DH} \\ \mathrm{EG} \end{Bmatrix} < \begin{Bmatrix} \mathrm{CH} \\ \mathrm{DF} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AD} \\ \mathrm{EF} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AC} \\ \mathrm{EH} \end{Bmatrix} < \{\mathrm{AE}\}$$

Floating-Figure 12. Subject 3 Perceptual Distance Constraints (Repaired Triads)

$$\begin{Bmatrix} \mathrm{AG} \\ \mathrm{BG} \\ \mathrm{CD} \\ \mathrm{CF} \\ \mathrm{DE} \\ \mathrm{DG} \\ \mathrm{FG} \\ \mathrm{GH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AB} \\ \mathrm{BH} \\ \mathrm{CE} \\ \mathrm{FH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{BF} \\ \mathrm{CH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AF} \\ \mathrm{EH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AH} \\ \mathrm{BE} \end{Bmatrix} < \begin{Bmatrix} \mathrm{BD} \\ \mathrm{EG} \end{Bmatrix} < \begin{Bmatrix} \mathrm{CG} \\ \mathrm{DF} \end{Bmatrix} < \begin{Bmatrix} \mathrm{BC} \\ \mathrm{DH} \\ \mathrm{EF} \end{Bmatrix} < \{\mathrm{AD}\} < \{\mathrm{AE}\} < \{\mathrm{AC}\}$$

Floating-Figure 13. Subject 4 Perceptual Distance Constraints (Repaired Triads)

$$\begin{Bmatrix} \mathrm{AF} \\ \mathrm{BE} \\ \mathrm{BF} \\ \mathrm{BH} \\ \mathrm{CD} \\ \mathrm{DE} \\ \mathrm{GH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{BD} \\ \mathrm{BG} \\ \mathrm{CE} \\ \mathrm{FH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AH} \\ \mathrm{EG} \\ \mathrm{FG} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AG} \\ \mathrm{DG} \\ \mathrm{EH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AB} \\ \mathrm{CG} \end{Bmatrix} < \{\mathrm{BC}\} < \{\mathrm{CH}\} < \begin{Bmatrix} \mathrm{CF} \\ \mathrm{DH} \end{Bmatrix} < \{\mathrm{EF}\} < \begin{Bmatrix} \mathrm{AE} \\ \mathrm{DF} \end{Bmatrix} < \{\mathrm{AC}\} < \{\mathrm{AD}\}$$

Floating-Figure 14. Subject 5 Perceptual Distance Constraints (Repaired Triads)

The similarity measure is simple: each distance group for the dimension is considered one at a time. For each distinct pair of items in that group, the program finds the indexes for those two items in the perceptual distance constraint clusters, and sums the difference between them.

$$\begin{Bmatrix} \mathrm{AG}\\ \mathrm{BD}\\ \mathrm{BF}\\ \mathrm{CD}\\ \mathrm{CE}\\ \mathrm{DG}\\ \mathrm{EH}\\ \mathrm{FH}\\ \mathrm{GH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{CH}\\ \mathrm{EG}\\ \mathrm{FG} \end{Bmatrix} < \begin{Bmatrix} \mathrm{CF}\\ \mathrm{DE} \end{Bmatrix} < \begin{Bmatrix} \mathrm{DH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{BH}\\ \mathrm{DF} \end{Bmatrix} < \begin{Bmatrix} \mathrm{BG}\\ \mathrm{EF} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AB}\\ \mathrm{BE} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AF}\\ \mathrm{BC} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AH}\\ \mathrm{CG} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AD} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AE} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AC} \end{Bmatrix}$$

Floating-Figure 15. Subject 6 Perceptual Distance Constraints (Repaired Triads)

$$\begin{Bmatrix} \mathrm{AB}\\ \mathrm{BG}\\ \mathrm{CE}\\ \mathrm{CH}\\ \mathrm{DF}\\ \mathrm{DG}\\ \mathrm{EF} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AG}\\ \mathrm{BD}\\ \mathrm{CF}\\ \mathrm{DE}\\ \mathrm{EH}\\ \mathrm{FG} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AD}\\ \mathrm{BF}\\ \mathrm{CD}\\ \mathrm{EG}\\ \mathrm{FH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AF}\\ \mathrm{BE}\\ \mathrm{CG}\\ \mathrm{DH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AE}\\ \mathrm{BC}\\ \mathrm{GH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AC}\\ \mathrm{BH} \end{Bmatrix} < \begin{Bmatrix} \mathrm{AH} \end{Bmatrix}$$

Floating-Figure 16. Generated Computed Distance Groups for ABGDFECH

$$\sum_{DG} \sum_{P_1, P_2} |Index(P_1) - Index(P_2)|$$

Each group in computed distance groups for the dimension represent pairs with the same distances between them; if that’s also true of the pairs in the perceptual distance constraint clusters, then the sum should be 0.

Subject	Topological	Score	CA Dimension	Score	rpg Dimension	Score
Subject 1:	ABGDFECH	[22]	ABGDHFEC	[43]	ABGDFHEC	[44]
Subject 2:	ECDBGHAF	[38]	EADGBCHF	[39]	EADGBCHF	[39]
Subject 3:	AHGFBCDE	[46]	AHGFBCDE	[46]	AHGFBCDE	[46]
	AHFGBCDE	[46]	AHGFBCDE	[46]	AHGFBCDE	[46]
Subject 4:	AGBFHCED	[32]	ABGHFEDC	[38]	ABGHFEDC	[38]
Subject 5:	AFHGBEDC	[48]	AFHGBECD	[48]	AFHGBECD	[48]
Subject 6:	AGDBFHEC	[35]	AGBFHDEC	[40]	ABFHGDEC	[40]

Floating-Figure 17. Topologically Discovered Dimensions Compared

Figure 17 shows a comparison of what I call the topologically found dimensions for the six subjects. The scores show how many of the triads are accommodated by the dimensions. The topologically found ones are never better at accommodating triads than the ones found by Alexander or by my program.

A final note is that even though topological sorting was the first thing I thought of for this problem when I first saw this paper, it seems that Alexander didn't consider it. My speculation is that he likely didn't know about topological sorting even though partial orders should have been familiar: the first publication I found about topological sorting algorithms was from 1960, and Alexander's paper was published also in 1960, submitted in 1959. For me it proved useful to find the inconsistencies in the triads and to repair them, but beyond that, the approach is not as effective as other sorts of search.

5 A TRIVIAL MISTAKE: NUMBER 1

In Fig. 7 (page 368) of Alexander's paper, he presents a simple diagram that shows for every subject the "triads satisfying betweenness on both similarity dimensions." Unlike some of the other errors in the paper, this table is easy to verify, though it might take some patience and perseverance. There is a trivial mistake in this table, and it is in a part that takes little effort for a casual reader to check. Here is that part:

Subject	Subject 1
Triads satisfying betweenness on both similarity dimensions	BDE BGE CED EFH*

The two similarity dimensions are ABGDHFEC & EDGCBFH. Looking, we can see that CED is in the first, but not the second. Recall that "satisfying betweenness" means that, for example, that CED is in the first because we can find it here: abg**D**hf**EC**, that is, in reverse order—because DEC is the same as CED as far as betweenness is concerned. The correct four triads in common are BDE, BGE, EDG, and EFH. (EFH* is marked with an asterisk (*) because EFH also appears in the preference order for Subject 1, which is CADGEBFH.) The corrected table is in Figure 18.

There are many possible explanations for this trivial mistake—typo, old version of the data, publisher error, transcription blunder (the G looked like a C and then the order of E and D was flipped). Nevertheless, it's a trivial error in the reporting of data in a journal.

6 A TRIVIAL MISTAKE: NUMBER 2

In Fig. 6 (page 367) of Alexander's paper, he presents the Kendall rank correlation coefficients (Kendall's τ coefficients) between each subject's preference order and his similarity dimensions.

Subject	Correlation of preference order with dimension 1	Correlation of preference order with dimension 2
Subject 1	0·00	+0·52
Subject 2	+0·29	−0·07
Subject 3	+0·14	0·00
Subject 4	0·00	0·00
Subject 5	−0·86*	+0·14
Subject 6	−0·36	+0·14

Fig. 6

The Kendall's τ coefficient between the preference order and Alexander's Dimension 1 is incorrect: it should be +0.21 instead of +0.29. The correct τ coefficients are shown in Figure 19.

The Kendall's τ coefficients for the 3 dimensions my program found are shown in Figure 20.

Subject 1	Subject 2	Subject 3	Subject 4	Subject 5	Subject 6
CADGEBFH	HGAFEBDC	BGHFDCEA	GCAHDEBF	DBCEGHFA	HDCBFGEA
BDE	ABC*	ABC	ABD	CBH	BDC
BGE	ABF	ABD	ABE	DBG*	BDE
CED-	ABH	ABE	AGD	DBH*	CEH
EDG+	ADC	ADE	AGE	EBG	CHF
EFH*	AGC	AFC	BGC	EBH	DHF
	AGH*	AGC	BGF	FHG*	EHF
	BDE	AGF	BGH		
	BGD	AHC	BHF		
	BGE	AHF	CDE*		
	CHF	BDE*			
	DGF	BGH*			
	DHF	CFG*			
	ECF	CFH*			
	ECH	DGH			
	EDF	EGH			
	EDG				
	EDH				
	EGF				
	EHF				

Floating-Figure 18. Triads Satisfying Betweenness on Both Similarity Dimensions

Subject	Dimension 1	Dimension 2	Dim 1 τ	Dim 2 τ	Preference Order
Subject 1:	ABGDHFEC	EDGCBFH	0.00	0.52	CADGEBFH
Subject 2:	ABFGHDCE	EADGBCHF	0.21	-0.07	HGAFEBDC
Subject 3:	AHGFBCDE	ABDEGHFC	0.14	0.00	BGHFDCEA
Subject 4:	ABGHFEDC	EDBGACHF	0.00	0.00	GCAHDEBF
Subject 5:	AFHGBECD	ECDFBHG	-0.86	0.14	DBCEGHFA
Subject 6:	AGBFHDEC	BDCGEHF	-0.36	0.14	HDCBFGEA

Floating-Figure 19. Corrected Kendall's τ Coefficients

Subject	Dimension 1	Dimension 2	Dimension 3	Dim 1 τ	Dim 2 τ	Dim 3 τ	Preference Order
Subject 1:	ABGDFHEC	AHCBDEGF	EFCDH	0.07	0.21	0.20	CADGEBFH
Subject 2:	AFHBGDCE	EADGBCHF	BCHGD	0.43	-0.07	0.00	HGAFEBDC
Subject 3:	AHGFBCDE	CEDBGHAF	BFC	0.14	-0.07	1.00	BGHFDCEA
Subject 4:	ABGHFEDC	AFHCGBDE	FGH	0.00	0.07	-0.33	GCAHDEBF
Subject 5:	AFHGBECD	CDEGHBF	ECF	-0.86	0.52	0.33	DBCEGHFA
	AFHGBEDC	DECGHBAF		-0.79	0.57		DBCEGHFA
Subject 6:	AGBFHDEC	DECGHBF	BFDHGCE	-0.36	0.05	0.24	HDCBFGEA
	AGBFHDEC	DCEGHBF	BFDHGCE	-0.36	0.14	0.24	HDCBFGEA

Floating-Figure 20. Kendall's τ Coefficients for rpg's Dimensions

7 DIMENSIONS EXPERIMENTS

Finally, I tried an oddball exercise: to program up several attempts to model the triad data in 3-dimensional space. In his paper, Alexander speculates there might be hidden factors or dimensions that account for the observed behavior of subjects. One can think of this as an n-dimensional factor space and the dimensions Alexander presents are projections of the subject's observations of that mental space onto axes. My idea was to see whether I could use ordinary 2- and 3-dimensional space as proxies for that mental space. That is, assign positions in space satisfying betweenness to try two things:

(1) see whether this approach could produce good relative closeness information from the triads
(2) see whether any derived dimension information was related to CA's idea of dimensions

Recall that given a triad ABC, the following perceptual distance constraints can be derived: $AB < AC$ and $BC < AC$. In all, each subject's triads imply 112 such constraints (numeric inequalities).

I did the experiments with both 2- and 3-dimensional space, but the 3-dimensional experiment produced better results and also corresponds to my other algorithm, which indicated one needs 3 dimensions to model the triads, not 2.

I tried two techniques that I've used before: a genetic algorithm and simulated annealing. Both are metaheuristic techniques. The genetic algorithm didn't work out super well; simulated annealing did better and was faster.

Basically, the approach is to place the 8 letters in a 3-dimension space: I set up a unit cube $(1 \times 1 \times 1)$ and placed the 8 letters in a uniform and neutral configuration. Namely, they were equally spaced from $(0, 0, 0)$ to $(1, 1, 1)$. (I also ran it with random initial positions—nothing much changed.)

Simulated annealing proceeds by "shaking" the configuration to try to maximize the number of triad betweenness rules the placement satisfies—that is, the number of numeric constraints that can be derived from a subject's triads. It's a randomized algorithm, which jiggles each letter's position, starting by sometimes permitting changes to make things worse and with larger possible jiggles, and gradually "cooling" the system by becoming less tolerant of bad moves and also by diminishing the size of possible jiggles. (https://en.wikipedia.org/wiki/Simulated_annealing)

I will show only results from Subject 1's repaired triads. The following is a pretty consistently achieved ordering of pairs of letters in various runs of the program, where pairs are sorted by the distance between them in 3-dimensional space:

```
CE, BD, FG, AB, DG, BG, DE, EF, CF, CH, CD, AG, DF, FH, AD, BF, BH, DH, GH, EH, EG, BE, CG, BC, AH, AF, AE, AC
```

This means that the pair `(C,E)` was the closest together and `(A,C)` the farthest apart. Look at Figure 10 for the topological clusters of distance; the above ordering is plausible.

Note: an ordering is ***consistent*** with a topologically derived set of constraints when the ordering could have been produced by those constraints; an ordering is ***plausible*** when it satisfies the following rules. Let $\{G_1, \ldots, G_n\}$ be a sequence of sets of objects—the topological clusters. Let

$$R_n = \bigcup_{i=1}^{n} G_i$$

Then an ordering $s_1, \ldots, s_m$ is plausible if

$$\forall 1 \leq i \leq m, s_i \in R_i$$

The run I show ended up satisfying all the (112) betweenness constraints. The placement of figures in 3-d space is given in Figure 21, and the distance pairs are given in Figure 22.

When the points are projected onto the x, y, & z axes, we get these (I placed Alexander's Dimension 2 under Dimension 3 in the display to make it easier to compare the results):

Subject 1 Repaired Triads

Source	Dimension 1	Dimension 2	Dimension 3
Program:	ABDHGCFE	GAFBDEHC	EDBCGAFH
rpg:	ABGDFHEC	AHCFEBDG	EDGCFH
Alexander:	ABGDHFEC	-	EDGCBFH

Not great, but not bad. I thought perhaps the 3-dimensional placements (the configuration of points) could be rotated so that the dimensions that either Alexander or I derived could be observed. But before diving into how to efficiently do simulated annealing on complex rotations, I thought I could add a few more constraints to the 112 to see whether the hoped-for dimensions would emerge. That was easy, so I tried it.

Figure	X	Y	Z
A	0.5367943821811844D0	0.07762039436192572D0	0.05314125201533126 4D0
B	0.2884034938576806D0	0.289161202882478 8D0	0.314089534092932 3D0
C	0.3169236072154282D0	0.943124097343058D0	0.778672358458693D0
D	0.16237877482932486D0	0.4909758360713165D0	0.38357411347553267D0
E	0.16092059222086902D0	0.9715895275236208D0	0.6546211741763004D0
F	0.5712076355291805D0	0.9438566605239186D0	0.24836154043820696D0
G	0.5184004370896663D0	0.7000324072128459D0	0.04387120281927699D0
H	0.8066435222789236D0	0.5913846398723082D0	0.7219416327698452D0

Floating-Figure 21. 3-d Coordinates for Figure Placement

The program was set up to search for placements of the eight figures so that not only the 112 constraints derived from the triads would be satisfied, but also that the projections would match ones I specified. I tried to satisfy these projections (my three dimensions for Subject 1):

ABGDFHEC, AHCFEBDG, EDGCFH

To my surprise, it did (see Figure 24 for the coordinates).

Subject 1 Repaired Triads with Dimension Targets

Source	Dimension 1	Dimension 2	Dimension 3
Targets:	ABGDFHEC	AHCFEBDG	EDGCFH
Program:	ABGDFHEC	AHCFEBDG	EDGCBFHA
Alexander:	ABGDHFEC	-	EDGCBFH

Notice that for Dimension 3, figures A & B were not part of the constraints, so their placement in the result is irrelevant. The resulting placement missed 3 of the 112 triad-derived constraints and got all 71 dimension-related constraints. However, it satisfied all 56 triads. I placed Alexander's Dimension 2 under Dimension 3 in the display above to make it easier to compare the results.

Pair	Distance
CE	0.20133533650261518D0
BD	0.24786989076388044D0
FG	0.3225758281229955D0
AB	0.41778170494998923D0
DG	0.5346532908758741D0
BG	0.5428919662329351D0
DE	0.5517772701212474D0
EF	0.5780583533032884D0
CF	0.588124533932942D0
CH	0.6056109447399359D0
CD	0.6200199824014394D0
AG	0.6227527477010929D0
DF	0.6249196095542088D0
FH	0.635566429230423D0
AD	0.6482558477578155D0
BF	0.7161874712341482D0
BH	0.7254343990459499D0
DH	0.7346098085381302D0
GH	0.7447603114296931D0
EH	0.7523602216637663D0
EG	0.757991185035489D0
BE	0.7732542314013713D0
CG	0.7997619777881148D0
BC	0.8026942536812287D0
AH	0.8854751310306376D0
AF	0.8886284385682102D0
AE	1.1411573133393D0
AC	1.1505803770163226D0

Floating-Figure 22. 3-d Pair Distances for Figure Placement

This result struck me as odd, so I tried to generate randomly chosen dimension targets. The program did so with few problems; to show that, I'll show you an extreme example. Just as my original program (Section 3) can find 3 dimensions directly from the triads by searching for dimensions that satisfy the most triads, that program can also find dimensions that satisfy the fewest triads. Here they are—I asked my program to find all the dimensions needed, and the best (worst?) it could do was two dimensions:

`DFHECABG, BGAECFHD`

I ran the 3-d positioning program several times (remember, it's a metaheuristic so doesn't always find a unique solution), and the results are in Figure 23.

Source	Dimension 1	Dimension 2	Dimension 3	Triad Score	Projection Score
Targets:	DFHECABG	BGAECFHD	-	-	-
Run 1:	DFHECABG	BGADECFH	ECDFHGBA	107/112	52/56
Run 2:	DECFHABG	BGAEFCDH	ABGDHFCE	110/112	50/56
Run 3:	DFHECABG	BGADECFH	CEFDHGBA	108/112	52/56

Floating-Figure 23. Subject 1 Repaired Triads with "Worst" Dimension Targets

In general, it seems that for any single target dimension, it's possible to place the figures in 3-d space so as to produce that dimension. To show this I did two final experiments searching for targeted dimensions, one for ABCDEFGH and one for Subject 1's preference order, CADGEBFH.

Figure	X	Y	Z
A	0.006020550238231197D0	0.3509839146064127D0	0.7217478811693852D0
B	0.4091897166956262D0	0.6555614218111547D0	0.3165928356968154D0
C	1.0	0.5001424649477755D0	0.25091117489209275D0
D	0.5548707617712716D0	0.7732892776881206D0	0.16267741587816695D0
E	0.8929860068010048D0	0.6437073102326842D0	0.14703490446516299D0
F	0.8204268446757994D0	0.6334789042163167D0	0.3732737321060554D0
G	0.5532106804739385D0	0.8302674572224099D0	0.22700597776065162D0
H	0.8262807580047175D0	0.3697082457029121D0	0.37627825771394696D0

Floating-Figure 24. Coordinates for Figure Placement & Projection Targets: ABGDFHEC, AHCFEBDG, EDGCFH

These are in Figures 25&26. I believe this shows that the triads simply provide inadequate constraints for the exercise of placing the figures in 3-d space to mean anything useful.

Source	Dimension 1	Dimension 2	Dimension 3	Triad Score	Projection Score
Target:	ABCDEFGH	-	-	-	-
Program:	ABCDEFGH	ABHDGCEF	CHEFDGBA	110/112	28/28

Floating-Figure 25. Subject 1 Repaired Triads with Dimension Target ABCDEFGH

Source	Dimension 1	Dimension 2	Dimension 3	Triad Score	Projection Score
Target:	CADGEBFH	-	-	-	-
Program:	CADGEBFH	ABHGDCEF	ABGDFEHC	111/112	28/28

Floating-Figure 26. Subject 1 Repaired Triads with Dimension Target CADGEBFH

Source	Dimension 1	Dimension 2	Dimension 3	Triads Covered	Repaired Triads Covered
Alexander:	ABGDHFEC	EDGCBFH	-	52	53
Run 1:	DFHECABG	BGADECFH	ECDFHGBA	51	52
Run 2:	DECFHABG	BGAEFCDH	ABGDHFCE	50	52
Run 3:	DFHECABG	BGADECFH	CEFDHGBA	51	52
abcdefgh:	ABCDEFGH	ABHDGCEF	CHEFDGBA	51	52
cadgebfh:	CADGEBFH	ABHGDCEF	ABGDFEHC	52	53
Other:	ABCDEFGH	BAGHDFEC	EFCGDBH	53	54
rpg 2-dims:	ABGDFHEC	AHCFEBDG	-	53	53
rpg 3-dims:	ABGDFHEC	AHCFEBDG	EDGCFH	56	56

Floating-Figure 27. Alexander’s Dimensions Compared

Figure 27 summarizes how well different proposed dimensions accommodate the triads for Subject 1. Looking at Figure 27 it’s hard to see that Alexander’s two dimensions explain observed behavior better than a variety of other ones.

8 REMARKS

Alexander concludes his paper as follows:

> *There are eight forms in the experiment. To account for a subject's sorting behaviour we therefore might need as many as seven dimensions—if his 'looking' were rich enough to involve seven ways at once. But we find that in every case two dimensions are enough to account for his entire behaviour.*

He goes on to discuss aesthetics, beauty, preferences, and linkages, but my interest in these notes is to determine whether the degrees of his thoroughness, meticulousness, and carefulness should give us cause to question his conclusions—whatever they may be. In this paper Alexander makes mistakes, makes claims with limited bases, and seems to explore only shallowly the data gathered in the experiments.

It turns out: three dimensions work better than two; his triads have consistency problems, some of which he seems to have noticed and some he didn't; there are at least two trivial errors he should have caught; the data he gathered provides too few constraints to produce dimensions, as can be seen by all the alternative proposed dimensions my programs came up with.

On the other hand, this paper makes no definitive claims other than the conducted experiments shed little or no explanatory light on the question of perceptual similarity. Nevertheless, more care would be welcome.

I readily grant that in 1959 or so, Alexander had access to computers likely not up to the tasks of helping analyze the data; that he was likely unaware of algorithms that could have been brought to bear; that the raw data was in a form that made clerical-like mistakes easy and likely; and that because the prospect of checking one's work was daunting, getting the work over with was attractive.

I said earlier that "Notes on the Synthesis of Form" contained some unsubstantiated implications, and that I have spent many weeks trying to replicate some of those implications. Likewise, I've explored his work with Bill Huggins on *local symmetries* or *sub-symmetries* in the early 1960s, and found that his analysis could have gone further were he to have had access to more powerful computers and more modern algorithms. Even though I generally believe his statements about wholeness, life, beauty, QWAN, process, and all that, I still believe when it came to the "mathy," algorithmic aspects of his work, he was a little careless and sloppy, and researchers following in his footsteps should beware of getting stuck in mud.

A "A RESULT IN VISUAL AESTHETICS" BY CHRISTOPHER ALEXANDER

For completeness I've included the full text of Alexander's paper. It starts on the next page.

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A RESULT IN VISUAL AESTHETICS

By CHRISTOPHER ALEXANDER
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Subjects were given eight forms and asked to sort them in a number of ways on the basis of overall similarity; they were also asked to state the order of their preferences among the forms. From the data generated, four tentative results emerged:

(1) By giving ourselves an appropriate verbal set, we can make ourselves see (and categorize) a group of forms in many different ways. It appears, however, that there is a natural way of categorizing them which is independent of any verbal set, and which depends on the formation of visual non-verbal concepts.

(2) Preference for the forms is dependent on the way the forms are seen—on the visual similarity dimensions. But the dependence is incomplete. It might perhaps be better called a linkage—comparable to the linkage between hue and brightness.

(3) The peculiarly weak nature of this linkage suggests the hypothesis that the beauty of a form cannot be explained in terms of any visible qualities or attributes that it has, but only in terms of the operations performed in the brain of the observer.

(4) Aesthetic discrimination is independent of all other kinds of perceptual discrimination.

I. Introduction

The way in which we look at things lies at the heart of visual aesthetics.

It is possible to look at a form in many different ways. We look at different aspects of the form—or we look for different things in the form. We may notice its plasticity, its movement, its simplicity, or any of a hundred other characteristics—characteristics that we can concentrate on one at a time.

To look at a form in a certain 'way' is to pay attention to (or to look at) a particular characteristic of the form. Now, we can set ourselves to look at any characteristic we wish. But what if we do not set ourselves deliberately at all? How do we look then? This is the problem we are interested in. We wish to find out:

(1) the 'ways' in which a subject naturally looks at forms;

(2) whether there is a connexion between his liking for the forms and the ways in which he looks at them;

(3) if there is such a connexion, its nature.

For the vague notion of a 'way' we shall substitute one more suitable for operational definition and analysis: the two-ended dimension. The situations where a subject pays attention to the plasticity, the movement, the simplicity of a form, we shall describe by saying that he is using the dimensions 'plastic-flat', 'dynamic-static', 'simple-complex'.

The first problem, then, is to find out which dimensions best describe the way a subject looks at forms. We could try several techniques.

(1) We might ask him to say which characteristics of the forms he paid attention to. And this would involve his stating, *in words*, the dimensions he believed himself to use. We could call them his introspective dimensions.

(2) We might present him with a long list of dimensions (plastic-flat, dynamic-static, simple-complex, open-closed, and many others), and ask him to place a number of forms on each of them (according to their plasticity, their movement, their sim-

plicity, their degree of closure, and so on). The semantic space set up by these dimensions could be factored and redefined in terms of a minimal set of dimensions (Osgood *et al.*, 1957; Tucker, 1955), which might then be said to describe the way the subject looked at the forms.

(3) More subtly still, we might use the following procedure, known as the method of triads. Three forms are shown to the subject, and he is asked which two are most alike. He is then asked to say in what respects the two are alike, and how the third one differs from them (Kelly, 1955). After he has given such answers for a number of triads we can see which aspects of the forms he pays most attention to when making his judgements, so we can construct (using, if possible, words that the subject himself has used) a set of dimensions that describe the way he looks at things (Henderson, Kates & Rohwer, 1959).

Yet all these methods are verbal ones—attempts made to fit verbal categories to visual phenomena. And while this is by no means impossible (it has been done with some success by critics, after all), there is a good deal of evidence to show that such attempts cannot get to the 'heart' of visual aesthetics.

Before we see why this is so, in full, let us consider a single incident that occurred during a triad experiment. A 12-year old girl was shown a Canaletto view of St Mark's Square, a Guardi view of the Grand Canal, and a line drawing of a single boat by Corot (all on postcards). Immediately she put the Venetians together as the two that were most alike—they were in fact extremely similar. But suddenly she remembered that she had to explain why or in what respects they were alike. And promptly she changed her mind, put the Guardi and the Corot together, and said, 'Those two, they've both got boats on them'. Visually this pairing was ridiculous. She had been forced by the demands of the experiment to group them in a way she had a word for. The distinguishing quality of the Venetian paintings was too hard for her to explain, though she could see it very well.

It is held by some psychologists that all our seeing is based on verbal categories (Brown, 1956; Whorf, 1941). That we only see what we have words for, that there is no visual concept formation, only verbal. And such a psychologist would say of the child, 'She actually saw like that; she saw in terms of the words she knew'.

We shall be able to show later that this view is false. That we can see independently of the words we know. For the present we shall simply remember that the child's *first* instinct was to put the Venetian paintings together.

Every experiment in visual aesthetics that deals with words is handicapped. People will not respond according to what they see, but according to the hopelessly inadequate vocabulary they have. And the results will be, just as the above one was, quite valueless from the point of view of visual aesthetics. If we are to achieve interesting results, we must let the subjects use their eyes. We must collect data which reflect only visual behaviour: we must find out the ways in which someone sees without letting him consider even, the words that described his ways of seeing.

II. The experiment

(1) *Stimuli*

The stimuli were 3 x 5 in. white filing cards, each with a single form drawn on it in black ink. About fifty forms were drawn quickly and freely; eight were then chosen from them. They were chosen for their mutual similarity, for their obviously set-like character, for the fact that they were variants on one another 'in several different directions'. The forms were named, at random, A, B, C, D, E, F, G, H (see Fig. 1). At no point of the experiment was the subject told the names of any of the forms—in case the names led him to make decisions on grounds connected with the letters. Subjects sometimes asked to see the forms upside down and from different angles. This was forbidden. If the subjects had seen the forms from several angles they might have noticed new aspects that would have led them to look at the forms in changed ways when it came to the experiment.

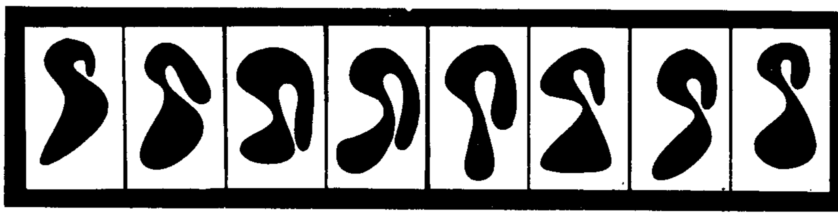

Fig. 1. The stimuli A, B, C, D, E, F, G, H.

(2) *Subjects*

There were six subjects, all well educated, all used to looking at things for pleasure, all between 20 and 30, three men and three women.

(3) *Procedure*

Two of the eight cards were chosen at random by the experimenter. They were laid in front of the subject, on a neutral ground. (The decision as to which card should go to the right and which to the left, was also made at random, and noted. During the repeat of the experiment, 24 hr. later, the pair was presented the opposite way round.) The remaining six cards were given to the subject, with the following instructions:

'Look at the six cards in your hand one by one, comparing them with the two cards on the table. For each one you must decide whether it looks more like the right hand or the left hand of the pair on the table. When you have decided, lay it on the table too, to the right or the left, according to the similarity you have observed.' (As the cards were laid down, they were kept at some distance from the pair already there; so that there should be no effects of visual proximity which might prejudice other decisions.) 'When you have placed all six cards in this way, make a check. If you wish to change your mind about any of them you may do so. You are under no compulsion to split the cards evenly; sometimes you may even want to put all six on the same side. Let it depend only on your feelings about the similarity of the forms.'

It was repeatedly made clear to the subject that he should try to be visually naïve, and should avoid making decisions on intellectual grounds. 'In the more difficult cases you will find yourself thinking hard about the decision. You may even feel that you could put a card on one side according to one criterion and on the other side according to some other criterion. If this does happen, stop thinking about it at once. Look away. When you look back, remember you are doing it on overall similarity, because forms look or feel alike. Forget that you are involved in an experiment, and imagine that an acquaintance has suddenly asked you, quite informally, "Which one does this look more alike?" Decide quickly; just like that.'

The visual interaction of some of the forms, when they were seen close together, seemed to have a disturbing effect also. To avoid it, where X and Y were forms on the table, and O one of the six forms being matched against them, subjects were given this instruction: 'O is to be compared with only one of X and Y at a time. Thus, while you look at O and X, keep Y covered. And when you look at O and Y, cover X. You are thus always looking at a pair, (O, X) or (O, Y). And you are to decide which makes the "closer" pair, (O, X) or (O, Y).' What was in fact being investigated here was the relation between a number of perceptual distances. If the subject found O more like X than Y, we may express this by saying that, for him, O is nearer to X than it is to Y; the perceptual distance OX is smaller than the distance OY.

As a further precaution the subject was asked never to put two cards really close together, but to keep several inches between them. When cards are very close to one another, the shape of the white between the two black forms becomes very important to the eye—and may upset the judgements made.

After the subject had made his six decisions they were written down like this:

X	Y
O	O
O	O
	O
	O

The subject went through this sorting procedure for each possible pair, i.e. twenty-eight times. The session took between an hour and an hour and a half. Twenty-four hours later all twenty-eight were repeated, in a different order, and with the pairs laid on the table which ever way round they had not been on the first occasion.

Subjects were quite often not consistent. For those cards which were placed in the pile they had been in 24 hr. before, the results were accepted, since here the subject seemed fairly certain, and there was very little doubt about his feelings on the matter. For those cards which were put in the pile they had not been in the first time, however, a closer scrutiny was necessary. Such inconsistency might have been caused in one of three ways.

(1) The most important cause, undoubtedly, lay in the very way the similarity judgements were made. It may be more likely that a subject puts O with X than that he puts it with Y. But if the decision is a hard one, as it often was, we shall at best observe this greater likelihood as a greater frequency. He will put O with X more often than with Y. And this can lead us to say that OX is smaller than OY by a narrow margin. (If the frequency turned out to be 50–50, we should have to call these two perceptual distances equal.) So if the subject put a card first in one pile, and then in the other, he was asked to sort it yet again—at once; and then again. Until it was clear which pile better pictured his feelings on the matter. (Each time he resorted it, the cards were rearranged so that he should not remember what he had done before.)

(2) The second possible reason for a subject's inconsistency lies in the arrangement on the table. The subject may be inclined to put cards to one side of his body rather than the other. Normally this factor will cancel out in the end, under the constant rearrangement. But if there is a card which he puts always to the same side, regardless of which pile lies on that side, one must assume him to be indifferent to the choice between piles and governed only by his left–right preference; and the perceptual distances will have to be taken as equal. (In all the experiments conducted, this happened only twice.)

(3) Thirdly, he may appear inconsistent because during the first time through he was unfamiliar with the cards, and has changed his mind now that he knows them better. This seemed to be the case a good deal of the time; during the second session

the subject was much surer of his attitude towards the cards than during the first. Where this was so, further sorting tended to support the subject's second decision rather than his first.

It is interesting to note that while subjects disagreed widely in their sorting on the first time through, there was strong agreement about the final versions. This suggests that the subject settled down gradually to his balanced judgement; and, what is more, it suggests that this balanced judgement corresponds only to stimulus characteristics, and independent of the particular subject whose judgement it is. Perhaps, if one were to stretch the experiment out even further, and let subjects settle down to their opinions still more slowly, we should find still greater agreement.

During the second session the subject was given a further task. Each time he finished with a pair, he was asked which member of the pair he preferred. Every subject thus made twenty-eight paired comparisons—which generated an order of preference over the eight cards. It was done during the second session only, so that the subject should be thoroughly familiar with the forms by the time he came to state his preferences. (In every case but one these paired comparisons were quite consistent and led to no intransitivities. The one subject (no. 7) who did produce intransitive results also produced very odd similarity data. He found all the tasks difficult, and said he could only do them on a consciously intellectual basis. His data have therefore not been included in the results of the experiment.)

Subjects were asked to make their preference judgements on grounds of shape alone —not to pay attention to associative overtones, but to judge the shape purely as a shape. Of course, they were not able to do this. The considerable disagreement illustrates this quite clearly. One always reads forms in a certain way. But they were trying to do it *this* way; and their judgements were governed by the feeling for form as much as possible.

Finally, when the whole experiment was over, the concept of a dimension was explained to the subject (though no specific cards were mentioned as illustrations, in case they biased his answer). The cards were now laid out in front of him, in random order, and he was asked what he thought he had been using as the bases for his similarity decisions, in spite of the fact that, at the time, he had been asked expressly to use no bases, principles, or criteria.

Each basis he gave—like longnose–shortnose—he was asked to illustrate with the most extreme examples. Thus he would be asked to point to the form with the longest nose and to that with the shortest. He was giving here his introspective dimensions; and most subjects gave three or four.

III. Tabulation and analysis of data

For each subject we now have the following data:

(1) Twenty-eight tables of the form

C	E
D	A
F	B
	G
	H

(2) A preference order generated by paired comparisons.

(3) A list of introspective dimensions given by the subject to account for his sorting behaviour.

Each table like

C	E
D	A
F	B
	G
	H

is in fact a condensed statement of six inequalities among the perceptual distances: for since F is put under C rather than under E, the table indicates that CF < EF; and five other facts of the same kind. Similarly the tables

C	F
B	A
D	G
E	H

and

E	F
C	A
D	B
G	H

tell us that CE < EF and CE < CF. We may combine the three inequalities to give CE < CF < EF. (When it happened—as it did about once for every subject—that the three inequalities were inconsistent, then one of them was reversed; whichever one the subject had been most uncertain of, whichever one he had changed his mind about most often.) The statement CE < CF < EF tells us that in the triad CEF the perceptual distance EF is the greatest of the three, so we write the triad ECF. The position of C between E and F is most important and will be referred to as this triad's betweenness.

Our data give us betweenness information of this kind for every one of the fifty-six triads; and for every subject (see Fig. 2). As has been shown in a recent paper (Hays, 1959; summarized in Coombs, 1958), this information as to betweenness may be used to generate orders which are closely similar to the dimensions obtained in factor analysis. (Factor analysis itself is not possible since there are no cardinal data available.) The following is a brief outline of the procedure used to generate these 'dimensions'. It is to be found, in full, together with its mathematical justification, in the papers mentioned. It is of course carried out separately for each subject's data.

(1) Select, by inspection of the data, that pair which seems to be most dissimilar (where the perceptual distance is the greatest), and use this pair as end-points of the first dimension.

(2) Order the remaining six letters between these end-points so as to accommodate as many of the fifty-six triads as possible. That is to say, construct that order which preserves betweenness for as many of the triads as possible.

(3) Repeat the above procedure for those triads not accommodated by the first dimension. If these are not enough to define a second dimension uniquely, construct that one which overlaps the first as little as possible. (It is inevitable, in spite of this, that there will be some triads whose betweenness is satisfied on both dimensions.)

In principle the extraction of dimensions should go on in this way until all the triads have been accommodated. But, in fact, no more than two dimensions were ever needed. All the triads, bar one or two, were accommodated by the first two

Triad	Subject 1	Subject 2	Subject 3	Subject 4	Subject 5	Subject 6
ABC	ABC	ABC	ABC	ABC	ABC	ABC
ABD	ABD	ABD	ABD	ABD	ABD	ABD
ABE	ABE	ABE	ABE	ABE	ABE	ABE
ABF	AFB	ABF	AFB	ABF	AFB	ABF
ABG	ABG	AGB	AGB	AGB	AGB	AGB
ABH	ABH	ABH	ABH	ABH	AHB	ABH
ACD	ADC	ADC	ADC	ADC	ACD	ADC
ACE	AEC	ACE	ACE	AEC	AEC	AEC
ACF	AFC	AFC	AFC	AFC	AFC	AFC
ACG	AGC	AGC	AGC	AGC	AGC	AGC
ACH	AHC	AHC	AHC	AHC	AHC	AHC
ADE	ADE	ADE	ADE	ADE	AED	ADE
ADF	ADF	ADF	AFD	AFD	AFD	AFD
ADG	AGD	AGD	AGD	AGD	AGD	AGD
ADH	ADH	AHD	AHD	AHD	AHD	AHD
AEF	AFE	EAF	AFE	AFE	AFE	AFE
AEG	AGE	AGE	AGE	AGE	AGE	AGE
AEH	AHE	AHE	AHE	AHE	AHE	AHE
AFG	AGF	AGF	AGF	AGF	AFG	AGF
AFH	AHF	AHF	AHF	AFH	AFH	AFH
AGH	AGH	AGH	AHG	AGH	AHG	AGH
BCD	BDC	BCD	BDC	BDC	BCD	BDC
BCE	BEC	BCE	BEC	BEC	BEC	BEC
BCF	BFC	BCF	BFC	BFC	CBF	BFC
BCG	BGC	CBG	CBG	BGC	BGC	CBG
BCH	BHC	BCH	CBH	BHC	CBH	CBH
BDE	BDE	BDE	BDE	BED	BED	BDE
BDF	BDF	DBF	DBF	DBF	DBF	DBF
BDG	BDG	BGD	DBG	BGD	DBG	BDG
BDH	DBH	DBH	DBH	DBH	DBH	BDH
BEF	BFE	EBF	EBF	EBF	EBF	BFE
BEG	BGE	BGE	EBG	EBG	EBG	BGE
BEH	BHE	EBH	EBH	BHE	EBH	BHE
BFG	BGF	FBG	BGF	BGF	FBG	BGF
BFH	BFH	BHF	BHF	BHF	FBH	BFH
BGH	GBH	BHG	BGH	BGH	BHG	BHG
CDE	CED	DCE	CDE	CDE	CDE	DCE
CDF	DCF	DCF	DCF	DCF	DCF	DCF
CDG	CDG	CDG	DCG	CDG	CDG	CDG
CDH	DCH	DCH	CDH	CDH	DCH	DCH
CEF	CEF	ECF	ECF	ECF	ECF	ECF
CEG	CEG	ECG	ECG	CEG	CEG	CEG
CEH	CEH	ECH	ECH	ECH	CEH	CEH
CFG	CFG	FCG	CFG	CFG	CFG	CFG
CFH	FCH	CHF	CFH	CFH	CHF	CHF
CGH	CHG	CHG	CGH	CHG	CGH	CGH
DEF	DEF	EDF	EDF	EDF	DEF	EDF
DEG	EDG	EDG	EDG	EDG	DEG	DGE
DEH	EDH	EDH	EDH	DEH	DEH	DEH
DFG	DGF	DGF	DGF	DGF	DGF	DGF
DFH	DFH	DHF	DHF	DFH	DHF	DHF
DGH	GDH	DGH	DGH	DGH	DGH	DGH
EFG	EFG	EGF	EGF	EGF	EGF	EGF
EFH	EFH	EHF	EFH	EHF	EHF	EHF
EGH	EHG	EGH	EGH	EHG	EGH	EHG
FGH	GFH	FHG	FGH	FGH	FHG	FHG

Fig. 2

dimensions extracted.* The dimensions are tabulated in Fig. 3. In Figs. 4*a* and 4*b* we have a visual presentation of the same material. Fig. 4*a* contains the six subjects' first dimensions, and 4*b* their second ones.

As we can see, the subject to subject agreement as to these dimensions is very high. The coefficient of concordance (Kendall, 1948) is 0·81 for the first dimension, and 0·65 for the second. Both these are above the 0·1 % level of significance. Since the agreement is so good, the modal dimensions are illustrated, for interest's sake, in Fig. 5. These are the principal 'ways' in which the average subject looked at the eight forms shown to him.

Subject	Dimension 1	Dimension 2	Introspective dimensions (in the order they were given)	Preference order
Subject 1	ABGDHFEC	EDGCBFH*	F–B Balanced—likely to topple F–H Sharp—round F–A Masculine—feminine B–A With weight—without weight F–D Pointing left—pointing right	CADGEBFH
Subject 2	ABFGHDCE	EADGBCHF	D–F Thin, linear—fat, solid H–F Round—angular C–A Long nose—short nose	HGAFEBDC
Subject 3	AHGFBCDE	ABDEGHFC	E–A Open—closed A–F Bird—snake F–E Solid—tottery	BGHFDCEA
Subject 4	ABGHFEDC	EDBGACHF	A–H Angular—round H–F Vertical—slanted G–A Indented—not indented	GCAHDEBF
Subject 5	AFHGBECD	ECDFBHG*	B–A Round—angular H–E Shaped—straight tail A–D Can't give it a name E–B Linear—triangular	DBCEGHFA
Subject 6	AGBFHDEC	BDCGEHF*	D–F Moving—massive base F–E Indented on the left—not indented E–F Unstable—stable	HDCBFGEA

* For three subjects the first dimension accommodated all triads containing A. In these cases the second dimension does not contain A.

Fig. 3

The thing that strikes us immediately about them is that they are almost impossible to name. We can describe them laboriously, of course, e.g. saying of the first one that the nose becomes less tiny and grows stronger, that the tail becomes less sharp and better formed, that the body becomes less slanted and more vertical, and of the second that there is a change from hanging to standing, that the form becomes less long and thin, and fuller-bodied, that there is a progressive change from instability to stability. But although we can *see* what is happening, as though a piece of rubber were being deformed, step by step, our words are hardly adequate.

The fact that we have only imprecise words for what we see here, is most important, since it brings us back to the point first mentioned in the introduction: that people

* Among the very few triads not accommodated by the first two dimensions, one or two are even inconsistent—Subject 1's AFB, for instance. Probably these minor vagaries are the result of the subject's indecision already discussed, and would be smoothed out if the subject were given still longer opportunity to reach consistent choices.

do not see only in terms of the words they have for a situation. Compare the introspective dimensions offered by the subjects, with the similarity dimensions we have extracted to describe what they actually did (Fig. 3). As we see, the degree to which subjects were verbally aware of what they were doing with their eyes and hands, is very limited. Moreover, while, as we have just seen, one subject's behaviour was

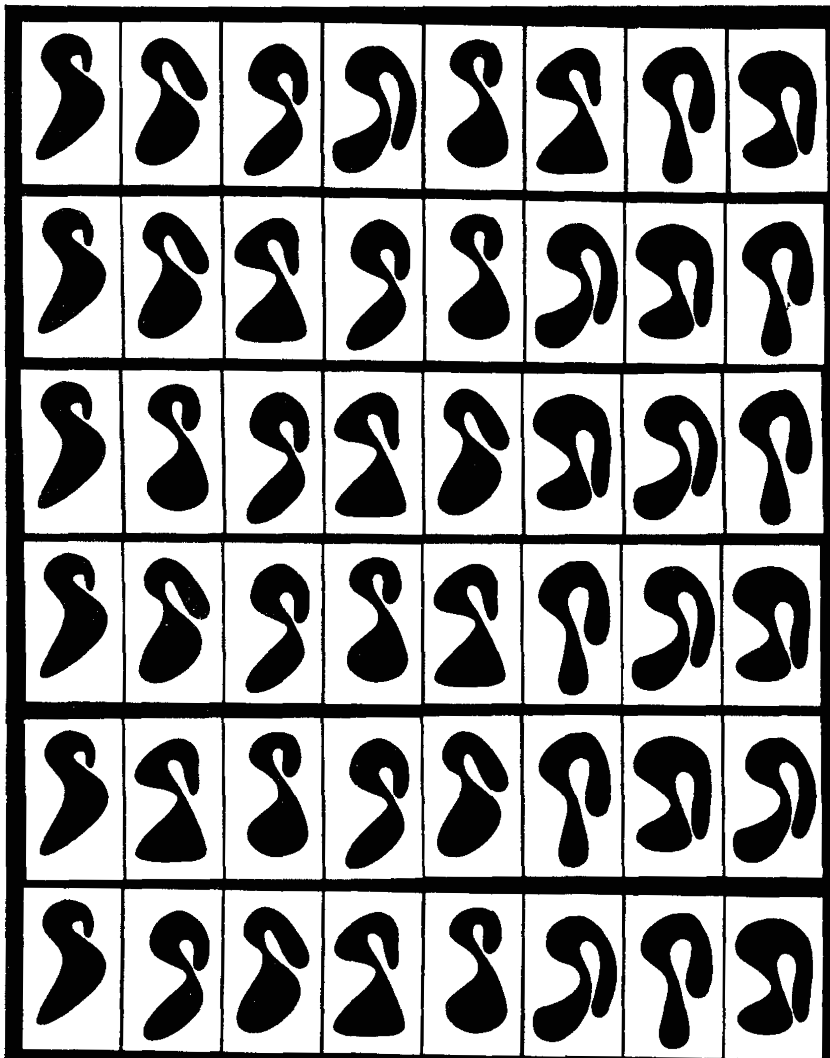

Fig. 4*a*. Dimension 1 for the six subjects.

much the same as another's, there is no such agreement from one subject to the next in the introspective dimensions offered. Indeed, these introspective dimensions seem to be irrelevant to the behaviour they were supposed to describe, They are connected, very certainly, with the subject's education, but unconnected, to any significant degree, with his visual behaviour.

Now for the central issue: to discover whether there is a connexion between a subject's liking for the forms and the ways in which he looks at them.

Preference orders generated by paired comparisons (the subjects' liking for the forms) are given in Fig. 3. Similarity dimensions (the subjects' ways of looking at the forms) are also given in Fig. 3. We shall test the possibility of a connexion in two different ways.

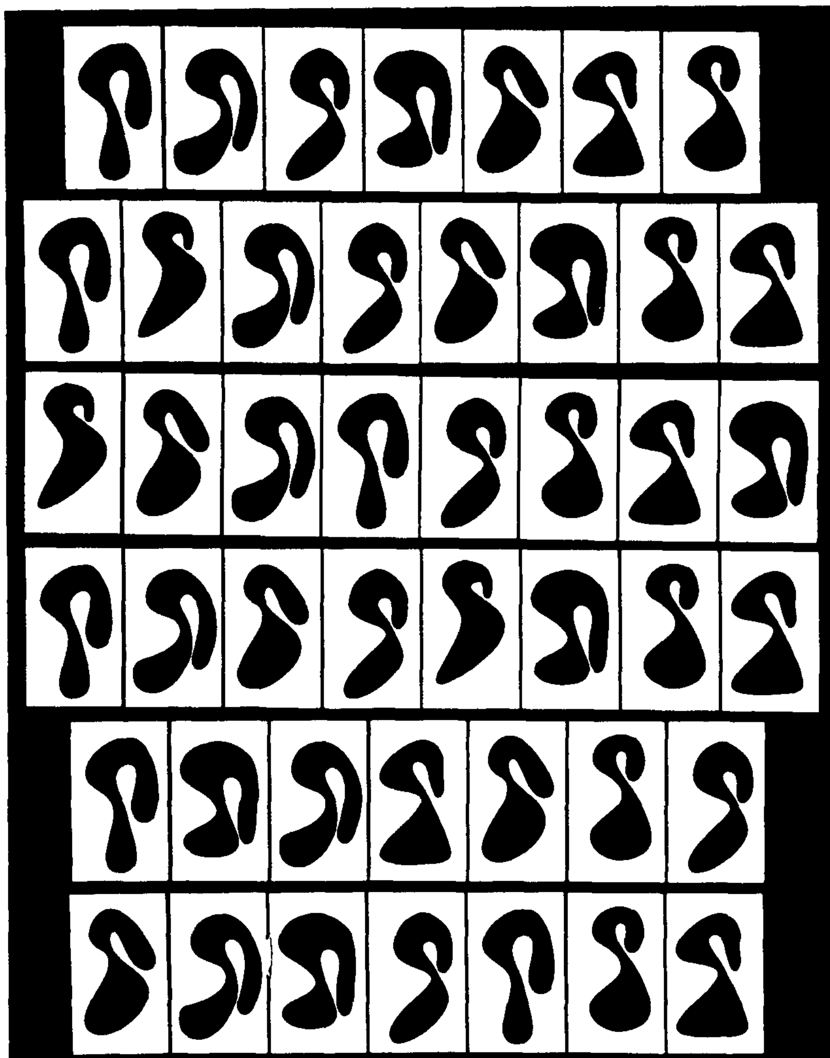

Fig. 4*b*. Dimension 2 for the six subjects.

(1) In the first test we shall examine each subject's preference order for its dependence on the similarity dimensions which describe that subject's behaviour. To do this we calculate the correlation between preference order and similarity dimensions. using the rank correlation coefficient τ (Kendall, 1948). These coefficients are presented in Fig. 6.

As we see, they are evenly distributed about zero, and only one (marked with a star) is above the 5 % level of significance. Since among twelve coefficients we should expect about one to reach this level, the test, points to no connexion whatever between the preference orders and the similarity dimensions.

(2) The second test is a more stringent one. For each subject there are certain triads which satisfy betweenness on both his similarity dimensions. These are listed in Fig. 7. Consider any such triad XOY (attached to a specific subject). O lies between X and Y on both his similarity dimensions. That is to say, in whichever of the two ways (or in whatever combination of them) he looks at the three forms, O will be intermediate between X and Y.

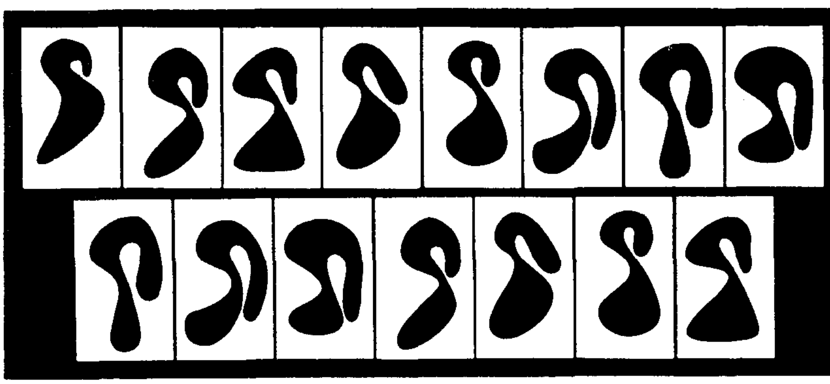

Fig. 5. Dimension 1 (modal); dimension 2 (modal).

Subject	Correlation of preference order with dimension 1	Correlation of preference order with dimension 2
Subject 1	0·00	+0·52
Subject 2	+0·29	−0·07
Subject 3	+0·14	0·00
Subject 4	0·00	0·00
Subject 5	−0·86*	+0·14
Subject 6	−0·36	+0·14

Fig. 6

If his preference order is positively connected with these dimensions, then, whatever the nature of the connexion, O should lie between X and Y on the preference order too.

In Fig. 7 we see how many of these triads do in fact satisfy betweenness on the preference orders: in all, ten out of fifty-nine. Now even if the triads were chosen at random we should expect a third of them to satisfy it; twenty out of fifty-nine, that is. So what do we make of our result? Certainly there is no positive connexion of the kind were were looking for. To establish that there would need to be twenty-seven out of fifty-nine for the 5 % level of significance. We have a situation which is quite the reverse, since the number quoted is significantly *less* than random (chi-square indicates a 0·7 % level of significance). This is a fact which demands explanation.

Subject	Subject 1	Subject 2	Subject 3	Subject 4	Subject 5	Subject 6	
Triads satisfying betweenness on both similarity dimensions	BDE	ABC*	ABC	ABD	CBH	BDC	
	BGE	ABF	ABD	ABE	DBG*	BDE	
	CED	ABH	ABE	AGD	DBH*	CEH	
	EFH*	ADC	AFC	AGE	EBG	CHF	
		AGC	AGC	BGC	EBH	DHF	
		AGH*	AHC	BGF	FHG*	EHF	
		BDE	ADE	BHF			
		BGD	AGF	BGH			
		BGE	AHF	CDE*			
		ECF	BDE*				
		ECH	BGH*				
		CHF	CFG*				
		EDF	CFH*				
		EDG	DGH				
		EDH	EGH				
		DGF					
		DHF					
		EGF					
		EHF					
							Total
Number of such triads	4	19	15	9	6	6	59
Number of these triads satisfying betweenness on the preference dimension (marked above with an *)	1	2	4	1	3	0	10

Fig. 7

IV. Discussion and conclusions

There are eight forms in the experiment. To account for a subject's sorting behaviour we therefore might need as many as seven dimensions—if his 'looking' were rich enough to involve seven ways at once. But we find that in every case two dimensions are enough to account for his entire behaviour. When asked to *say* how they looked at the forms, subjects all put forward more than two ways, it is true. But this belief in the subtlety of their looking must have been largely wishful thinking—for what they actually *did* could be described with only two dimensions.

There must be no misunderstanding at this point. The introspective dimensions offered by the subject are not meaningless. If he wants to, he can look at the forms in these ways—indeed, he *can* look at the forms in any way he pleases, for whatever verbal dimensions we make up he can set himself to place the forms along them (Osgood *et al.*, 1957). But what is essential is what he actually *did*. And what he did can be described with only two dimensions. If some third 'say' had been important to him, this fact would have been reflected in his sorting behaviour; and we should need a third dimension to account for it. (Very likely, if there had been more than eight forms, we should have needed more than two dimensions to describe his behaviour.) What is clear at any rate is that looking, as a process of categorization, is simpler than we think.

The dimensions that we extract are not verbal ones, but visual. They have no names and, directly, we cannot discuss them. But we can see them with our own eyes: we can see that something is changing from one end of the dimensions to the other, but we are hard put to it to give the 'something' a name. Often, in fact,there are no ready words to describe the ways in which we look at forms. What about the subject's introspective dimensions—the ways in which he *thought* he had been looking at the forms? It turns out that although there is some agreement between these introspective (verbal) dimensions, and the visual ones we have extracted from his sorting behaviour, this agreement is very limited, a phenomenon which reminds us of the little girl and the Venetian paintings, but far more conclusive. We do not see only according to the way we think. On the contrary, we do not have words for what we do with our eyes.

We cannot describe our visual behaviour introspectively; and it seems that it makes good sense to refer to visual concepts which are non-verbal. What is more, while the subjects all have the same visual concepts, they have widely different verbal ones. They differ in their descriptions of their own seeing behaviour, even though the behaviour itself is much the same for all of them, a fact which does not speak well for the view that seeing is based on a learned net of language. On the contrary, the most plausible explanation is that we all share the same sort of perceptual apparatus, but have all been brought up differently, and have different words for similar visual phenomena. Our verbal concepts are largely personal—our visual ones are not.

It is one of the gifts of the great critic that, by coining words or putting old words to new uses, he can name dimensions we all use with our eyes but which we have not yet been able to name for oursleves (Wölfflin, 1915). And the painter's gift is greater still, for he makes us see (use) dimensions that we not only have no word for, but do not even know with our eyes.

Is there a connexion between a subject's liking for the forms, and the ways in which he looks at them? There is no significant correlation between the preference orders and the similarity dimensions. Nor do triads which satisfy betweenness on both similarity dimensions satisfy it on the preference dimensions, as they should if there were a positive connexion of any sort between them. We must say, therefore, that there is no positive connexion. However, the number of such triads satisfying betweenness on the preference dimension is smaller than we should expect from chance. It looks, indeed, as though a triad satisfying betweenness on both similarity dimensions will tend not to satisfy it on the preference dimension—a sort of negative interaction.

Perhaps we can illuminate this by restating it. Forms that lie at the centre of both similarity dimensions tend to lie toward the ends of the preference dimension. When a form lies at the centre of both our similarity dimensions we either like it or dislike it, but are not indifferent to it. We feel strongly about such a form. And conversely, forms at the ends of similarity dimensions tend to be neither especially beautiful, nor especially not so.

It is difficult to make much of this information. Art critics feel something similar, perhaps, when they say that a form containing divergent elements is particularly good if these elements are successfully unified, but if the contrast between them is not resolved, the form is particularly bad. Forms which lie at the centre of similarity dimensions may be said to contain diverging elements (namely, the two ends of the dimension). And the contrast between these elements will be resolved or unresolved, the form good or bad, but not indifferent.

So it appears that there is a connexion after all. Forms which lie at the centre of both similarity dimensions tend to induce strong feelings in the subject. So much we can say. What we do not know, however, is whether these strong feelings will be favourable or unfavourable; whether the forms will be liked or disliked. While the similarities seen do restrict the preference order, they do not determine it. The preference dimension is dependent on the similarity dimensions—but the dependence is incomplete. It might, perhaps, be better called a linkage—comparable to the linkage between hue and brightness (yellows tend to be brighter than blues or reds, though hue and brightness are independent otherwise). It has been shown that attributes may often be linked to one another in very subtle ways, without being what we should normally call dependent (Stevens, 1934). In this case the weakness of the linkage is twofold:

(1) The connexion is not complete and would better be called a tendency.

(2) There is an ambiguity at the crucial point of the connexion, for we cannot predict whether the strong feeling induced will be liking or dislike.

It is well known that it is difficult to explain the beauty of forms, and that those explanations which *are* offered, are at best partial ones. Perhaps the difference we see between good forms and bad is essentially irreducible to any other differences we see, just as the difference we see between two hues of equal brightness and saturation is irreducible. For what does a successful explanation of visual quality depend upon? That our aesthetic discrimination can be made dependent on the other discriminations of which we are capable. That there is some unambiguous mapping from the other qualities our eyes allow us to pick out, onto the aesthetic one.

Yet the experiment suggests that just this is not the case. The mapping is many-many, or, as we put it, no more than a linkage. The aesthetic explanation it allows can be no more powerful therefore, than explanations of the difference between blue and yellow couched in terms of brightness could be.

This hypothesis puts no restriction on explanations which are possible in terms of experimental psychology or physiology. But it does suggest that explanations in terms of other visible qualities are essentially restricted by the weakness of the linkage.

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